Solutions

61. *Find all real solutions of the following system of equations:*

$$\begin{aligned}\sqrt{x^2+y^2+6x+9}+\sqrt{x^2+y^2-8y+16} &= 5,\\ 9y^2-4x^2 &= 60.\end{aligned}$$

(50th Catalonian Mathematical Olympiad)

Solution 1 by Eloi Torrent Juste, AULA Escola Europea, Barcelona, Spain. First we observe that points (x, y) that satisfy the first equation are those that the sum of their distances to $A(-3, 0)$ and $B(0, 4)$ is equal to 5. Moreover, if a point P lies out of the segment AB then $AP + PB > AB = 5$. This let us to conclude that points (x, y) solution of the system must lie on AB. The equation of AB is $y = \dfrac{4}{3}x + 4$ or $\left(x, \dfrac{4}{3}x + 4\right)$ with $-3 \leq x \leq 0$. Substituting these values in the second equation, yields

$$9\left(\frac{4}{3}x+4\right)^2-4x^2=60 \Leftrightarrow x^2+8x+7=0$$

with roots $x = -7$ and $x = -1$. Since only the second lie in $[-3, 0]$, then the unique solution of the given system is $(-1, 8/3)$.

Solution 2 by Arkady Alt, San Jose, California, USA. Squaring both sides of the equation $\sqrt{x^2+y^2+6x+9}+\sqrt{x^2+y^2-8y+16}=5$ we have

$$\left(\sqrt{x^2+y^2+6x+9}+\sqrt{x^2+y^2-8y+16}\right)^2=25 \Leftrightarrow 4x-3y+12=0$$

Then, from

$$\left.\begin{array}{c}4x-3y+12=0\\ 9y^2-4x^2=60\end{array}\right\} \Leftrightarrow \left.\begin{array}{c}3y=4x+12\\ (4x+12)^2-4x^2=60\end{array}\right\} \Leftrightarrow \left.\begin{array}{c}3y=4x+12\\ 12\,(x+7)\,(x+1)=0\end{array}\right\}$$

we obtain

$$(x,y)=\left(-1,\frac{8}{3}\right)$$
$$(x,y)=\left(-7,\frac{16}{3}\right)$$

By substitution immediately follows that only $(x, y) = \left(-1, \dfrac{8}{3}\right)$ satisfies the given system and it is the desired solution. □

Also solved by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain.

62. *Let P be an interior point to an equilateral triangle ABC. Draw perpendiculars PX, PY and PZ to the sides BC, CA and AB, respectively. Compute the value of*

$$\frac{BX+CY+AZ}{PX+PY+PZ}$$

(First BARCELONATECH MATHCONTEST 2014)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. First let us denote the side length of the triangle by a. The area of the triangle can be calculated in two ways and we get $\frac{\sqrt{3}}{2}a^2 = a(PX + PY + PZ)$, hence

$$PX + PY + PZ = \frac{\sqrt{3}}{2}a \tag{1}$$

On the other hand.

$$aBX = \overrightarrow{BC} \cdot \overrightarrow{BP}, \quad aCY = \overrightarrow{CA} \cdot \overrightarrow{CP}, \quad aAZ = \overrightarrow{AB} \cdot \overrightarrow{AP}$$

hence

$$\begin{aligned} a(BX + CY + AZ) &= \overrightarrow{BC} \cdot \overrightarrow{BP} + \overrightarrow{CA} \cdot \overrightarrow{CP} + \overrightarrow{AB} \cdot \overrightarrow{AP} \\ &= \underbrace{(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB})}_{\vec{0}} \cdot \overrightarrow{BP} + \overrightarrow{CA} \cdot \overrightarrow{CB} + \overrightarrow{AB} \cdot \overrightarrow{AB} \\ &= \overrightarrow{CA} \cdot \overrightarrow{CB} + \overrightarrow{AB} \cdot \overrightarrow{AB} \\ &= a^2 \frac{1}{2} + a^2 \end{aligned}$$

so

$$BX + CY + AZ = \frac{3}{2}a \tag{2}$$

Thus, from (1) and (2) we get

$$\frac{BX + CY + AZ}{PX + PY + PZ} = \sqrt{3}.$$

□

Solution 2 by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. Joining A, B, C with P we obtain three pairs of right triangles: AZP, AYP; BZP, BXP and CXP, CYP. If a is the length of the side of $\triangle ABC$, then on account of Pithagoras theorem, we have

$$AZ^2 + ZP^2 = (a - CY)^2 + PY^2$$

$$BX^2 + PX^2 = (a - AZ)^2 + PZ^2$$

$$CY^2 + PY^2 = (a - BX)^2 + PX^2$$

Developing and adding up, yields

$$BX + CY + AZ = \frac{3a}{2}$$

On the other hand the sum of the areas of triangles APB, BPC, CAP is the area of $\triangle ABC$. That is,

$$\frac{a(PX + PY + PZ)}{2} = \frac{a^2\sqrt{3}}{4} \Leftrightarrow PX + PY + PZ = \frac{a\sqrt{3}}{2}$$

From the preceding immediately follows that

$$\frac{BX + CY + AZ}{PX + PY + PZ} = \sqrt{3}$$

and we are done. □

Solution 3 by Arkady Alt, San Jose, California, USA. Let $a = BC = CA = AB, x = BX, y = CY, z = AZ, u = PX, v = PY, w = PZ$ and $h = \dfrac{a\sqrt{3}}{2}$ be height of the equilateral triangle ABC, then

$$[ABC] = [PBC] + [PCA] + [PAB] \Leftrightarrow \frac{ah}{2} = \frac{au}{2} + \frac{av}{2} + \frac{az}{2} \iff u + v + w = h$$

and

$$\frac{BX + CY + AZ}{PX + PY + PZ} = \frac{x+y+z}{u+v+w} = \frac{x+y+z}{h} = \frac{2\,(x+y+z)}{a\sqrt{3}}$$

Applying Pythagorean theorem to chain of right triangles $\triangle PXB, \triangle PBZ, \triangle PZA, \triangle PAY, \triangle PXB, \triangle PYC, \triangle PCX$, we obtain

$$\begin{cases} u^2 + x^2 = w^2 + (a-z)^2 \\ w^2 + z^2 = v^2 + (a-y)^2 \\ v^2 + y^2 = u^2 + (a-x)^2 \end{cases}$$

Adding all equations we get

$$\sum_{cyc} \left(u^2 + x^2\right) = \sum_{cyc} \left(w^2 + (a-z)^2\right) \Leftrightarrow 3a^2 = 2a\,(x+y+z)$$

S0, $x + y + z = \dfrac{3a}{2}$ and, therefore, $\dfrac{BX + CY + AZ}{PX + PY + PZ} = \sqrt{3}$. □

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

63. *How many ways are there to weigh of* 31 *grams with a balance if we have* 7 *weighs of one gram,* 5 *of two grams, and* 6 *of five grams, respectively?*

(Training Catalonian Team for OME 2014)

Solution 1 by José Luis Díaz-Barrero BARCELONA TECH, Barcelona, Spain. The required number is the number of solutions of $x + y + z = 31$ with

$$x \in \{0,1,2,3,4,5,6,7\}, \quad y \in \{0,2,4,6,8,10\}, \quad z \in \{0,5,10,15,20,25,30\}$$

We claim that the number of solutions of this equation equals the coefficient of x^{31} in the product

$$(1 + x + x^2 + \ldots + x^7)\,(1 + x^2 + x^4 + \ldots + x^{10})\,(1 + x^5 + x^{10} + \ldots + x^{30})$$

Indeed, a term with x^{31} is obtained by taking some term x from the first parentheses, some term y from the second, and z from the third, in such a way that $x+y+z = 31$. Each such possible selection of x, y and z contributes 1 to the considered coefficient of x^{31} in the product. Since,

$$\begin{aligned}&(1 + x + x^2 + \ldots + x^7)\,(1 + x^2 + x^4 + \ldots + x^{10})\,(1 + x^5 + x^{10} + \ldots + x^{30}) \\ &= 1 + x + \ldots + 10x^{30} + 10x^{31} + 10x^{32} + \ldots + x^{46} + x^{47},\end{aligned}$$

then the number of ways to obtain 31 grams is 10, and we are done. □

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. We are looking for the number of triplets (k, l, m) such that

$$k \in \{0, \ldots, 7\}, \; l \in \{0, \ldots, 5\}, \; m \in \{0, \ldots, 6\}, \; k + 2l + 5m = 31.$$